Indian Statistical Institute,Bangalore Centre

B.Math.(Hons.)II Year-2013-14,First Semester

Optimization

Back Paper Exam 08 Jan 2014, 10am - 1pm.

Instructor: P.S.Datti Max.Marks: 100

NOTE: Solve all the questions. WRITE NEATLY.

1. If A is a non-singular matrix, show that its LU decomposition is unique. **(4)**

2. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{pmatrix}$. Obtain the LU decomposition of A and use it to solve the system $Ax = b$, where $b = \begin{pmatrix} 14 \\ 18 \\ 20 \end{pmatrix}$. **(6+6)**

3. Suppose A is a real $m \times n$ matrix of rank n.
   (a) Explain in detail how to obtain a QR decomposition of A, where Q is an $m \times n$ matrix satisfying $Q^tQ = I$ and R is an upper triangular square matrix of order n with positive diagonal elements. **(8)**
   (b) Show that the above QR decomposition is unique. **(4)**

4. Suppose A, B are real $m \times n$ matrices such that $A^tA = B^tB$. Show that there is an $m \times m$ orthogonal matrix U such that $A = UB$. **(6)**

5. Suppose A is a real $m \times n$ matrix and P is the orthogonal projection onto $im(A)$. If $b \in \mathbb{R}^m$, show that
$$\|Ax - b\| \geq \|b - Pb\|$$
for all $x \in \mathbb{R}^n$, with equality if and only if $Ax = Pb$. **(6)**

6. Suppose A is a non-negative square matrix of order n. For $x \in \mathbb{R}^n$, $x \geq 0$, $x \neq 0$, define
$$r_x = \min_{1 \leq i \leq n} \left\{ \frac{(Ax)_i}{x_i} : x_i > 0 \right\}$$
$$R_x = \max \{\rho \geq 0 : Ax \geq \rho x\}$$
Show that $r_x = R_x$. **(6)**

7. Explain in detail why the $\lim_{k \to \infty} A^k$ exists, where
$$A = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{pmatrix},$$

and find the limit. **(4+4)**

8. Reduce the following minimization problem to the standard form linear programming:

$$\text{minimize} \quad |x| + |y| + |z|$$

subject to

$$x + y \leq 1 \text{ and } 2x + z = 3.$$

(4)

9. State the Minkowski-Farkas Lemma and prove it using duality theorem of linear programming. **(2+4)**

10. Consider an LPP in standard form:

$$\text{minimize } c^t x, \text{ subject to } Ax = b,\ x \geq 0.$$

Here A is a real $m \times n$ matrix of rank m.

(a) Define an optimal feasible solution and a basic optimal feasible solution of the above LPP. **(2+2)**

(b) If there is an optimal feasible solution, show that there is also a basic optimal feasible solution. **(8)**

11. Given that the following LPP:

$$\text{maximize} \quad \frac{3}{4}x_1 - 20x_2 + \frac{1}{2}x_3 - 6x_4$$

subject to

$$\begin{aligned} \frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 &\leq 0 \\ \frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 &\leq 0 \\ x_3 &\leq 1 \end{aligned}$$

$$x_i \geq 0, \ i = 1, 2, 3, 4.$$

has an optimal solution $(1, 0, 1, 0)$.

(a) Write down the corresponding dual problem. **(4)**

(b) Solve the dual problem by simplex method. **(10)**

(c) Verify the duality theorem of linear programming for the above situation. **(2)**

12. Using the duality theorem of linear programming, show that the system " $Ax > 0$" has a solution if and only if " $A^t u = 0,\ u \geq 0 \Rightarrow u = 0$." **(4+4)**